of the method to modeling errors has been quantified. The method has a reasonable level of robustness to errors in the low mode. The robustness to errors in the second mode varies greatly and is poor for large ranges of the system parameters. These results indicate that robustness techniques for multimode systems that are based on limiting the derivative of the final state with respect to system parameters may not be effective for all parameter values.

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# Range-Rate Control Algorithms and Space Rendezvous Schemes

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## Introduction

SPACECRAFT rendezvous, containing a maneuvering spacecraft and a target spacecraft, consists of two successive phases: long-distance navigation and short-distance homing. In the navigation phase, the orbit parameters as well as the positions of the two spacecraft may differ greatly. The main task of this phase is an impulsive orbit transfer, usually controlled from the ground station. This is an open-loop control. Both orbit determination and impulse generation techniques are applied. The achievable control accuracy of these techniques is rather below that required for the subsequent docking of the two spacecraft. Therefore, the goal of the navigation phase is to bring the maneuvering spacecraft to within a specified vicinity of the target spacecraft (for example, less than 100 km) in the same orbit plane from which the homing phase or terminal rendezvous begins.

In the homing phase, interest is in the motion of the maneuvering spacecraft relative to the target spacecraft. Control of the motion is carried out in such a way that the maneuvering spacecraft moves closer toward the target spacecraft along a stable trajectory and finally stops near it.

Clohessy and Wiltshire[1] developed a control method based on the analytical solution to the linearized differential equations of the relative motion in circular orbits. As the method is open loop, it is very sensitive to the modeling error as well as to the error of control impulse generation. Whereas the method can ensure the end position of the maneuvering spacecraft, it may not be able to ensure a desirable intermediate course of the trajectory in the motion. Another widely used control method is based on the proportional/parallel navigation law.[2] This method is based solely on kinematics and does not take into account the specific dynamics of the motion, which results in degradation of the control performance.

In Refs. 3 and 4, a control method, called the range-rate control algorithm (RRCA), and its modification, that is, the omnidirectional range-rate control algorithm (ODRRCA), have been proposed for terminal rendezvous control. These control algorithms, based on nonlinear dynamical system theory, more clearly display the dynamics of the motion such as the fixed point (equilibrium state) and its stability. They also demonstrate very good performance and implementability.

However, all of these methods are only effective for the target spacecraft in a circular orbit. In the case of an elliptic orbit, the dynamics of the motion becomes more complicated. Instead of a fixed point, there is a limit cycle (periodic motion) in the system. This Note addresses RRCA and ODRRCA, applied to space rendezvous in an elliptic orbit, and suggests three generalized schemes for the terminal rendezvous control.

## Equations of Motion and Control Algorithms

The equations of the coplanar rendezvous motion, written in a reference system fixed with the line of sight of the spacecraft, are given as[3]

$$\ddot{D} - D(\dot{v} + \dot{\varphi})^2 + \left(\mu / R_T^3\right) D(1 - 3\sin^2\varphi) = a_S \tag{1}$$

$$D(\ddot{v} + \ddot{\varphi}) + 2\dot{D}(\dot{v} + \dot{\varphi}) - 3\left(\mu / R_T^3\right) D \sin\varphi\cos\varphi = a_\varphi \tag{2}$$

where $D$ is the distance between the two spacecraft, $\varphi$ is the direction angle or phase angle of the line of sight measured from the local horizontal, $v$ is the true anomaly, $R_T$ is the orbital radius of the target spacecraft, $\mu$ is the gravitational coefficient, and $a_S$ and $a_\varphi$ are forced-acceleration components of the spacecraft. As $\dot{D} \le 0$ in the rendezvous motion, $2\dot{D}\dot{\varphi}$ in Eq. (2) is zero or negative damping, which indicates an instability of the motion. Therefore, the control algorithms, however proposed, should stabilize the motion.

Two control algorithms proposed for the terminal rendezvous are as follows.

### RRCA and the Controlled Motion

There is a reference range rate $\dot{D}_r$ in RRCA,[3] defined as

$$\dot{D}_r = \frac{k\dot{v} + k_1\dot{\varphi}}{\dot{v} + \dot{\varphi}}\dot{v}D, \qquad k \in [-0.75, 0], \qquad k_1 > 0$$

where $k$ and $k_1$ are selectable parameters of the RRCA. The control system will make $\dot{D}$ equal to $\dot{D}_r$ by regulating only the propulsion $a_S$ of the maneuvering spacecraft aligned with the direction of the line of sight (the so-called in-line propulsion) in the form

$$a_S = j(\dot{D}_r - \dot{D}), \qquad j > 0, \qquad a_\varphi = 0$$

The control error ratio $(\dot{D}_r - \dot{D})/|\dot{D}_r|$ is so small[3] that the controlled range rate $\dot{D}$ and, therefore, the distance $D$ can be approximated by

$$\dot{D} = \frac{k\dot{v} + k_1\dot{\varphi}}{\dot{v} + \dot{\varphi}}\dot{v}D \tag{3}$$

The directional motion of $\varphi$ can be obtained by substituting Eq. (3) into Eq. (2) as follows:

$$\ddot{\varphi} + 2k_1\dot{v}\dot{\varphi} - 1.5\left(\mu / R_T^3\right)\sin 2\varphi = -2k\dot{v}^2 - \ddot{v} \tag{4}$$

### ODRRCA and Controlled Motion

ODRRCA consists of two parts as follows.

1) In-line propulsion $a_S$ is the same as for RRCA; thus, the distance motion is given by Eq. (3).

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2) A transverse propulsion $a_\varphi$ is defined as

$$
\begin{aligned}
a_\varphi &= \left(\mu/R_T^3\right)D(a\sin 2\varphi + b\cos 2\varphi)\\
a &= -1.5 + 2k\sin 2\varphi_d - 0.5q\cos 2\varphi_d \qquad (5)\\
b &= 2k\cos 2\varphi_d + 0.5q\sin 2\varphi_d
\end{aligned}
$$

where $q > 0$ is selectable and $\varphi_d$ is selectable from the interval $(0, 2\pi)$. Incorporating Eqs. (3) and (5) into Eq. (2) yields the directional angle motion of ODRRCA as

$$
\ddot{\varphi} + 2k_1\dot{v}\dot{\varphi} + c\left(\mu/R_T^3\right)\cos(2\varphi - 2\varphi_d - \alpha) = -2k\dot{v}^2 - \ddot{v} \qquad (6)
$$

$$
c = \sqrt{4k^2 + 0.25q^2}, \qquad \sin\alpha = q/2c, \qquad \cos\alpha = -(2k/c)
$$

There are some relations for the orbital elements

$$
\dot{v} = \sqrt{\mu/p^3}(1 + e\cos v)^2, \qquad \ddot{v} = -2(\mu/p^3)e\sin v(1 + e\cos v)^3
$$

$$
R_T = p/(1 + e\cos v), \qquad (1 + e\cos v)^{-1} \approx 1 - e\cos v \quad (\text{for } e < 0.3)
$$

which may be used to replace $t$ by $v$ and reduce Eqs. (4) and (6) accordingly into

$$
\varphi'' + 2[k_1 - e\sin v(1 - e\cos v)]\varphi' - 1.5(1 - e\cos v)\sin 2\varphi = -2k + 2e\sin v(1 - e\cos v) \qquad (7)
$$

$$
\varphi'' + 2[k_1 - e\sin v(1 - e\cos v)]\varphi' + c(1 - e\cos v) \times \cos(2\varphi - 2\varphi_d - \alpha) = -2k + 2e\sin v(1 - e\cos v) \qquad (8)
$$

where the prime indicates the derivative with respect to $v$. The differential equations (7) and (8) are nonlinear and dissipative. In the case of a circular orbit $\dot{v} = \text{const}$, and these equations have stable fixed points determined for RRCA as

$$
\varphi_0 = -\tfrac{1}{2}\sin^{-1}\left(\tfrac{4}{3}k\right) + (n + 0.5)\pi, \qquad n = 0, 1
$$

and for ODRRCA as

$$
\varphi_0 = \varphi_d + n\pi, \qquad n = 0, 1
$$

In the case of an elliptical orbit, the coefficients of Eqs. (7) and (8) are $2\pi$ periodical so that these equations may have a periodic solution (limit cycle). Reference 5 gives a numerical iterative method of computing the limit cycles and analyzing their stability. Based on the method, a set of stable limit cycles for different $e$ has been computed[6] (Figs. 1 and 2). In general, the location, size, configuration, and stability of the limit cycle depend on the values of the parameters $k$, $k_1$, $\varphi_d$, and $q$, and especially on the eccentricity $e$. There is a critical $e_c$ depending on the parameters $(k, k_1, \varphi_d, q)$, above which the limit cycle becomes unstable. On each cycle, there are two extremals $\varphi_{\min}$ and $\varphi_{\max}$. The periodic motion of $\varphi$ occurs within the interval $[\varphi_{\min}, \varphi_{\max}]$. Each stable limit cycle (including a fixed point) has a domain of attraction. Any motion of $\varphi$ within the domain of attraction will converge with the limit cycle. Accordingly, the maneuvering spacecraft's trajectory will come to and remain in a sector on the orbital plane, formed by the two angles $\varphi_{\min}$, $\varphi_{\max}$. The sector is called the angular corridor of the rendezvous trajectory.

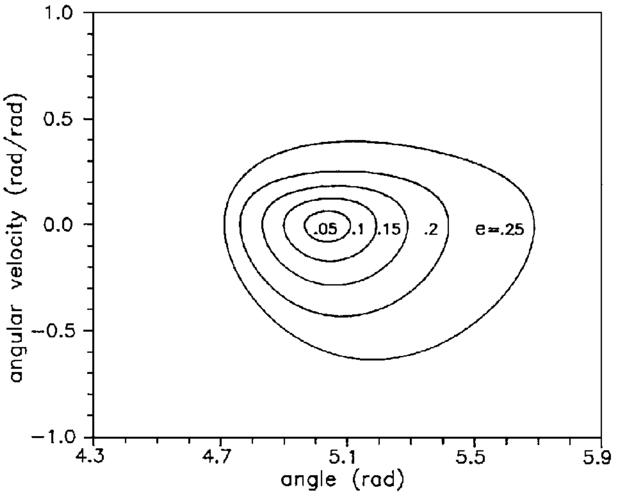


**Fig. 1 Limit cycles in phase plane ($\varphi$, $\varphi'$): RRCA.**

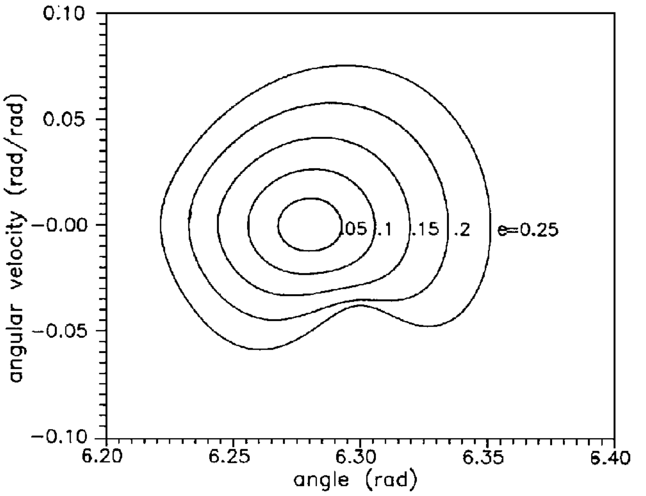


**Fig. 2 Limit cycles in phase plane ($\varphi$, $\varphi'$): ODRRCA.**

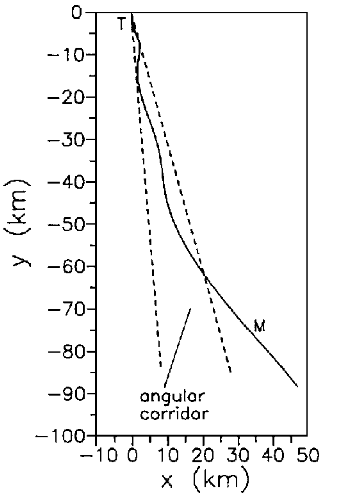


**Fig. 3 Rendezvous trajectory: RRCA-based scheme.**

## Three Proposed Schemes for Terminal Rendezvous Control

### RRCA-Based Scheme

In this scheme only the in-line propulsion is required, carrying out the RRCA algorithm. The angular corridor is situated either in the interval [90 deg, 135 deg] or in the interval [270 deg, 315 deg] in the orbital plane.[3] Figure 3 shows a simulated trajectory of the maneuvering spacecraft. This simulation as well as others have been conducted with all nonlinear terms in the equations of motion and for an elliptic orbit of $e = 0.1$. The advantage of this approach is a simple propulsion scheme. The disadvantage is that the angular corridor can be placed only in the two narrow segments mentioned.

### ODRRCA-Based Scheme

There are two components, that is, the in-line propulsion and transverse propulsion, working in the ODRRCA principle. The angular corridor can be placed anywhere in the entire orbital plane. In many cases, the maneuvering spacecraft will come to the target spacecraft from behind ($\varphi_d = 0$). Figure 4 shows such a trajectory of the maneuvering spacecraft. The ODRRCA-based scheme has the advantage that besides an arbitrariness of the angular corridor's position, there are four selectable parameters $k$, $k_1$, $\varphi_d$, and $q$ that allow a good performance of the scheme. The range of angle $\varphi$ is

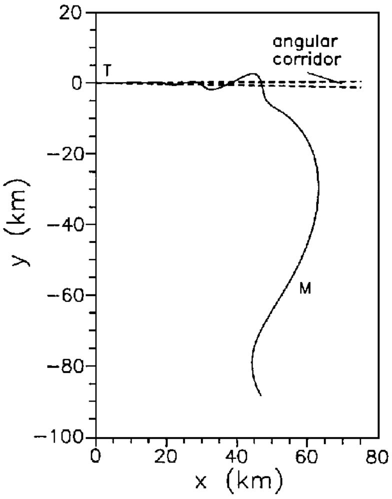


**Fig. 4 Rendezvous trajectory: ODRRCA-based scheme.**

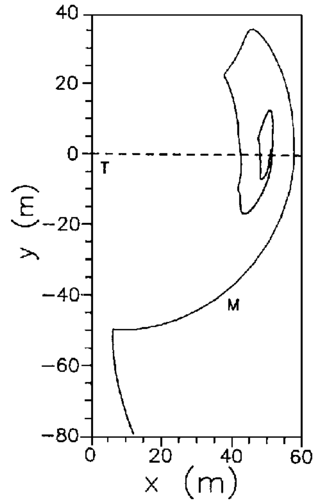


**Fig. 5 Round flight trajectory: combined scheme.**

also small. A cost to pay is the relatively complex two-component propulsion system needed.

**Combined Scheme**

This scheme is a combination of the two preceding ones. The entire rendezvous operation is divided into two steps. In the first step, which is called the main part and starts at the beginning of the terminal rendezvous, the RRCA-based scheme is applied until the maneuvering spacecraft arrives at a designed end distance (for example, 50 m) from the target spacecraft. At this moment the second step, called the round flight, begins. In this step, the ODRRCA-based scheme takes the place of the RRCA-based scheme. By properly setting the sign and magnitude of $k$ near zero, the distance between the two spacecraft does not change significantly. Rather, the maneuvering spacecraft flies by the target spacecraft and finally stops at a desirable position, determined by the parameter $\varphi_d$. The value of $\varphi_d$ is assigned by the mission designer. The trajectory in the main part of the combined scheme is the same as shown in Fig. 3, and Fig. 5 shows the round flight part of the trajectory. The combined scheme preserves the advantages of the two previous schemes with fewer complications of the propulsion system.

## Conclusions

The RRCA and the ODRRCA are nonlinear control algorithms. They are, in particular, suitable for the control of the relative motion between two spacecraft such as the tethered satellite maneuver, space rendezvous, etc. Based on these algorithms, three generalized control schemes for terminal rendezvous have been proposed: the RRCA-based scheme, the ODRRCA-based scheme, and the combined scheme. Various rendezvous missions in both circular and elliptic orbits can be carried out by these schemes with variable-thrust or fixed-thrust engines. For a typical terminal rendezvous started at a separation distance of 100 km, the fuel consumption of the maneuvering spacecraft with initial mass of 7800 kg is about 175 kg, and the time required to complete the rendezvous is about 220 min. The distance between two spacecraft as well as the distance rate decrease in an exponential manner, while the rendezvous trajectory can be placed in a desirable position to meet the mission requirements.

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# Optimized TRIAD Algorithm for Attitude Determination

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## I. Introduction

WHEN the components of two abstract vectors are given in two different coordinate systems, it is possible to find the orientation difference between the two systems. In particular, we can easily find the transformation matrix from one coordinate system to the other. TRIAD[1,2] is an algorithm that does just that. The process of finding the matrix using TRIAD is as follows. Let $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ denote the column matrices whose elements are, respectively, the components of the two abstract vectors when resolved in one coordinate system (typically a body frame), and let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ denote the column matrices whose elements are, respectively, the components of the abstract vectors when resolved in the other coordinate system (typically a reference frame). The algorithm calls for the computation of the following column matrices in the body frame:

$$\boldsymbol{r}_1 = \boldsymbol{w}_1/|\boldsymbol{w}_1| \tag{1a}$$

$$\boldsymbol{r}_2 = \frac{(\boldsymbol{r}_1 \times \boldsymbol{w}_2)}{|\boldsymbol{r}_1 \times \boldsymbol{w}_2|} \tag{1b}$$

$$\boldsymbol{r}_3 = \boldsymbol{r}_1 \times \boldsymbol{r}_2 \tag{1c}$$

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